

BIFURCATION VALUES OF  $C^\infty$  FUNCTIONS

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ABSTRACT. We show how one can use a trivialization of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  on fibers of some function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  to construct a trivialization of  $f$  in  $\mathbb{R}^n$ . Additionally we adopt a method for trivialising functions which satisfy the  $\rho_0$ -regularity condition to the case of functions defined on hypersurfaces of the form  $M = g^{-1}(0)$ .

## INTRODUCTION

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a polynomial. It is well-known that there exists a finite set such that  $f$  is a  $C^\infty$  fibration over the complement of this set. The smallest such a set is called the set of *bifurcation values* of  $f$  and denoted by  $B(f)$ . It contains the set of critical values  $K_0(f)$  and the set of *bifurcation values at infinity*  $B_\infty(f)$  of  $f$ . Finding an effective description of the set  $B_\infty(f)$  is still an open question. However we can approximate  $B_\infty(f)$  using different supersets. The most popular one uses the so-called Malgrange condition.

We say that  $f$  satisfies the *Malgrange condition* in  $\lambda \in \mathbb{R}$  if there exists a neighborhood  $U$  of  $\lambda$  and constants  $\delta, R > 0$  such that

$$(M) \quad \forall_{x \in f^{-1}(U)} \quad \|\nabla f(x)\| \|x\| \geq \delta \quad \text{for} \quad \|x\| \geq R.$$

The set  $K_\infty(f)$  of all values which do not satisfy Malgrange's condition is called the set of *asymptotic critical values* of  $f$  i.e.

$$K_\infty(f) := \{\lambda \in \mathbb{R} \mid \exists_{(x_k) \subset \mathbb{R}^n} \|x_k\| \rightarrow \infty, f(x_k) \rightarrow \lambda, \|x_k\| \|\nabla f(x_k)\| \rightarrow 0\}.$$

Jelonek and Kurdyka in [JK1] gave an effective characterization of this set. Moreover in [JK2] they showed how to proceed when the polynomial  $f$  is defined on an algebraic set (see also [Je]).

In this paper we show how one can use a trivialization of a smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  on fibers of some other smooth function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  to construct a desired trivialization of  $f$  in  $\mathbb{R}^n$ . More precisely, we introduce a

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notion of  $(g, S)$ -Malgrange condition; for a smooth function  $g$  and an open set  $S \subset \mathbb{R}$  we say that  $f$  satisfies the  $(g, S)$ -Malgrange condition in  $\lambda$  if

- (1)  $\nabla g(x) \neq 0$  near  $f^{-1}(\lambda)$  outside some compact set
- (2)  $g^{-1}(S)$  contains all fibers of  $f$  near  $\lambda$
- (3) for any  $s \in S$  the function  $f|_{g^{-1}(s)} : g^{-1}(s) \rightarrow \mathbb{R}$  satisfies the Malgrange condition in  $\lambda$

(compare to the definition in Section 3). The set of all  $\lambda$  that don't satisfy the  $(g, S)$ -Malgrange condition we will denote by  $K_\infty^{(g, S)}(f)$ . Our aim is to prove  $B(f) \subset K_\infty^{(g, S)}(f) \cup K_0(f)$  and therefore

$$B(f) \subset \bigcap_{(g, S)} K_\infty^{(g, S)}(f) \cup K_0(f)$$

(see Theorem 3.1 and Remark 3.5). It is worth noting that in (3) of the definition of  $(g, S)$ -Malgrange condition we allow different constants  $\delta_s$  (in inequality (M)) for different  $s \in S$ . Therefore our method may be applied even if  $\inf\{\delta_s \mid s \in S\} = 0$  (i.e. when the standard Malgrange condition fails). This often leads us to more sharp approximation of  $B(f)$  than in the classical method (see Example 3.6 and Example 3.7).

The other popular approach to finding the bifurcation values uses a critical set  $\mathcal{M}_a(f)$  of the map  $(f, \rho_a)$  where  $\rho_a$  is the Euclidean distance function from a fixed point  $a \in \mathbb{R}^n$ . We can define the set of asymptotic  $\rho_a$ -nonregular values as

$$S_a(f) := \{\lambda \in \mathbb{R} \mid \exists_{(x_k) \subset \mathcal{M}_a(f)} \|x_k\| \rightarrow \infty, f(x_k) \rightarrow \lambda\}.$$

It has been proven in [Ti], [DRT], [DT] that  $B_\infty(f) \subset S_a(f)$  for any  $a \in \mathbb{R}^n$ , thus in particular

$$B_\infty(f) \subset S_\infty(f) := \bigcap_{a \in \mathbb{R}^n} S_a(f).$$

In the second part of this paper we will show how to get a similar result when  $f$  is defined on the manifold of the form  $g^{-1}(0)$ . We will introduce an analog of the condition used in [NZ1] and [NZ2] that allows us to trivialize function  $f$  (see Theorem 4.1 and Remark 4.5).

## 1. AUXILIARY RESULTS

In this section we collect some useful facts about differential equations, which we will use later in this article.

Let  $M$  be a smooth  $m$ -dimensional manifold. We denote by  $T_x M$  the tangent space to the manifold  $M$  at a point  $x \in M$  and  $TM := \bigcup_{x \in M} T_x M$ . Let  $W : M \rightarrow TM$  be a smooth vector field on  $M$ .

**Lemma 1.1.** *Let  $\phi : (t_0, \xi] \rightarrow M$  be a solution of the system of differential equations  $x' = W(x)$ . Assume that there exists a sequence  $t_k \in (t_0, \xi)$ ,  $k \in \mathbb{N}$ , such that  $\lim_{k \rightarrow \infty} t_k = t_0$  and  $\lim_{k \rightarrow \infty} \phi(t_k) = x_0 \in M$ . Then there exists  $\lim_{t \rightarrow t_0} \phi(t) = x_0$  and  $\phi$  is not the maximal solution to the left.*

*Proof.* Let  $\varphi = (\varphi_1, \dots, \varphi_m) : U \rightarrow A \subset \mathbb{R}^m$  be a map in a neighborhood  $U \subset M$  of the point  $x_0$ . Choosing a subsequence if necessary, we may assume that  $x_k := \phi(t_k) \in U$  for  $k \in \mathbb{N}$ . Then we can write

$$W(x) = \sum_{i=1}^m W_i(x) \frac{\partial}{\partial \varphi_i} \quad \text{for } x \in U,$$

where  $W_i : U \rightarrow \mathbb{R}$  for  $i = 1, \dots, m$ .

Let  $y_0 := \varphi(x_0)$  and  $y_k := \varphi(x_k)$  for  $k \in \mathbb{N}$ .

Let  $\alpha \in \mathbb{R} \cup \{-\infty\}$  be a minimal number for which  $\phi((\alpha, t_1]) \subset U$ . Put  $I = (\alpha, t_1]$ . Denote  $\phi_\varphi := \varphi \circ \phi : I \rightarrow A$  and  $\phi_{\varphi_i} := \varphi_i \circ \phi$  for  $i = 1, \dots, m$ .

Then we have  $\phi'(t) = \sum_{i=1}^m \phi'_{\varphi_i}(t) \frac{\partial}{\partial \varphi_i}$  for  $t \in I$  and

$$\sum_{i=1}^m \phi'_{\varphi_i}(t) \frac{\partial}{\partial \varphi_i} = \sum_{i=1}^m W_i(\phi(t)) \frac{\partial}{\partial \varphi_i}.$$

Therefore

$$(1) \quad \phi'_{\varphi_i}(t) = W_{\varphi^{-1}}^i(\phi_{\varphi_i}(t)) \quad \text{for } t \in I \quad i = 1, \dots, m,$$

where  $W_{\varphi^{-1}}^i(y) := W_i \circ \varphi^{-1}(y)$  for  $y \in A$ .

To complete the proof we need the following well known fact.

**Property 1.2.** *There exists an interval  $J$  such that  $t_0 \in J$  and a neighborhood  $\Gamma \subset \mathbb{R} \times A$  of  $(t_0, y_0)$  such that for any  $(t', y') \in \Gamma$  every maximal solution  $\gamma$  of sytem (1) that passes through  $(t', y') \in \Gamma$  is defined at least on  $J$  and the graph of  $\gamma|_J$  is contained in some rectangle  $T \subset \mathbb{R} \times A$ .*

By choosing a subsequence of the sequence  $(t_k, y_k)$ , we may assume that  $(t_k, y_k) \in \Gamma$  for  $k \in \mathbb{N}$  and  $\xi > t_k > t_l > t_0$  for  $k < l$ .

From Property 1.2 we have that  $\alpha \leq t_0$ . Indeed, otherwise there exists  $t' \in (t_0, t_1)$  such that  $\phi(t') \notin U$ , hence there exists a maximal solution to the left  $\widehat{\phi}_\varphi : \widehat{I} \rightarrow A$  of the system (1) such that  $\widehat{\phi}_\varphi$  goes through  $\Gamma$  and  $t_0 \notin \widehat{I}$  which contradicts Property 1.2.

Therefore  $(t_0, t_1] \subset I$  and  $\phi_\varphi = \phi_\varphi^*|_I$  where  $\phi_\varphi^* : I^* \rightarrow A$  is a maximal solution to the left of the system (1) and  $t_0 \in I^*$ . Consequently

$$\lim_{t \rightarrow t_0} \phi(t) = \lim_{t \rightarrow t_0} \varphi^{-1}(\phi_\varphi^*(t)) = \lim_{t \rightarrow t_0} \varphi^{-1}(y_0) = x_0,$$

which completes the proof of Lemma 1.1.  $\square$

**Lemma 1.3.** *Let  $\phi : (\alpha, \beta) \rightarrow M$  be a maximal solution of the system  $x' = W(x)$ . For every compact set  $K \subset \mathbb{R} \times M$  there exist  $\alpha^*, \beta^* \in \mathbb{R}$  such that the graphs of  $\phi|_{(\alpha, \alpha^*)}$  and  $\phi|_{(\beta^*, \beta)}$  are disjoint with  $K$ .*

*Proof.* Let  $A := \{t \in (\alpha, \beta) \mid (t, \phi(t)) \in K\}$ . If  $A = \emptyset$  then as  $\alpha^*, \beta^*$  we may choose arbitrary numbers from  $(\alpha, \beta)$ . Now assume that  $A \neq \emptyset$  and let  $\alpha^* = \inf A$ ,  $\beta^* = \sup A$ . Observe that  $\alpha < \alpha^*$  and  $\beta^* < \beta$ . Indeed, if  $\alpha = \alpha^*$  then there exists a sequence  $(t_k)_{k=1}^\infty$  such that  $(t_k, \phi(t_k))$  converges to a point in  $K \subset \mathbb{R} \times M$ . This contradicts Lemma 1.1. Analogously we prove that  $\beta^* < \beta$ .  $\square$

## 2. THE MALGRANGE CONDITION ON MANIFOLDS

Let  $f, g \in C^\infty(\mathbb{R}^n)$ . We will assume that  $\nabla g(x) \neq 0$  for  $x \in g^{-1}(0)$ . Denote  $M := V(g) = g^{-1}(0)$  and  $f_M := f|_M$ . Consider the following vector field  $\nabla f_M : M \rightarrow \mathbb{R}^n$

$$(2) \quad \nabla f_M(x) := \nabla f(x) - \frac{\langle \nabla f(x), \nabla g(x) \rangle}{\|\nabla g(x)\|^2} \nabla g(x), \quad x \in M.$$

Geometrically  $\nabla f_M(x)$  is the projection of  $\nabla f(x)$  onto the tangent space  $T_x M$ .

A value  $\lambda \in \mathbb{R}$  is called a *regular value* of  $f_M$  if  $\nabla f_M(x) \neq 0$  for  $x \in f_M^{-1}(\lambda)$ . The set of all values  $\lambda$  that are not regular we will denote by  $K_0(f_M)$ .

We say that  $f_M$  satisfies the *Malgrange condition* over  $U \subset \mathbb{R}$  if there exist constants  $\delta, R > 0$  such that

$$\forall_{x \in f_M^{-1}(U)} \quad \|\nabla f_M(x)\| \|x\| \geq \delta \quad \text{for} \quad \|x\| \geq R.$$

We say that  $f_M$  satisfies the *Malgrange condition* in  $\lambda \in \mathbb{R}$  if there exists a neighborhood  $U$  of  $\lambda$  such that  $f_M$  satisfies the Malgrange condition over  $U$ . We will denote by  $K_\infty(f_M)$  the set of all  $\lambda$  that do not satisfy the Malgrange condition.

It is well known that the Malgrange condition allows us to integrate  $\nabla f_M(x)/\|\nabla f_M(x)\|^2$  field and get the trivialization of  $f_M$  (see [Ra] and [Je]). More precisely, we have

**Theorem 2.1.** *Let  $\lambda \in \mathbb{R}$  be a regular value of  $f_M$ . If  $f_M$  satisfies the Malgrange condition in  $\lambda \in \mathbb{R}$  then there exists a neighborhood  $U$  of  $\lambda$  such that  $f_M|_{f_M^{-1}(U)}$  is a  $C^\infty$  trivial fibration on  $M$ .*

Immediately from Theorem 2.1 we get

**Remark 2.2.**  $B(f_M) \subset K_\infty(f_M) \cup K_0(f_M)$ .

Note that if  $f_M$  satisfies the Malgrange condition, it does not follow that  $f|_{M^\varepsilon}$  satisfies it also, where  $M^\varepsilon = g^{-1}((-\varepsilon, \varepsilon))$ . In other words, one can not

trivialise  $f_M$  using field  $\nabla f_M$  in a neighbourhood  $M^\varepsilon$  of  $M$  as it is shown by the following example.

**Example 2.3.** Let  $f, g \in \mathbb{R}[x, y]$  be polynomials defined as  $g(x, y) := y$ ,  $f(x, y) := x - x^3 y^2$ . Put  $M = g^{-1}(0)$ . Obviously  $\nabla f_M = [1, 0]$  and  $f$  satisfies Malgrange condition on  $M$ . Consequently  $f_M$  is a trivial fibration on  $M$ . On the other hand, for any  $\varepsilon > 0$ , denoting  $M_t := g^{-1}(t)$  for  $t \in (-\varepsilon, \varepsilon)$  we have  $\nabla f_{M_{g(x,y)}}(x, y) = (1 - 3x^2 y^2, 0)$  for  $(x, y) \in M^\varepsilon$ . Moreover, for

$$(x_n, y_n) = \left( \frac{n\varepsilon}{2}, \frac{2}{\sqrt{3n\varepsilon}} \right), \quad n \in \mathbb{N}$$

we have  $\|\nabla f_{M_{g(x,y)}}(x_n, y_n)\| = 0$  for  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \|(x_n, y_n)\| = \infty$ , and  $(x_n, y_n) \in M^\varepsilon$  as  $n \rightarrow \infty$ . So we can not use the trivialization method without restricting the field  $\nabla f_M$  to the manifold  $g^{-1}(0)$ .

### 3. THE MALGRANGE CONDITION ON FIBERS

In this section we will present a method of using fibers of some function  $g$  to construct a trivialisation of a function  $f$ .

Let  $f, g \in C^\infty(\mathbb{R}^n)$ . Let  $D \subset \mathbb{R}^n$  be an open set and  $\nabla g(x) \neq 0$  for  $x \in D$ . As in (2) we can define  $\nabla f_{g^{-1}(g(x))}(x)$  for  $x \in D$ . Put

$$\nabla_g f(x) := \nabla f_{g^{-1}(g(x))}(x) \quad \text{for } x \in D.$$

Geometrically  $\nabla_g f(x)$  is the projection of  $\nabla f(x)$  onto the tangent space at  $x$  of the fibre  $g^{-1}(g(x))$ .

Denote  $\Xi = \{(g, S) \mid g \in C^\infty(\mathbb{R}^n), S \text{ open in } \mathbb{R}\}$  and let  $(g, S) \in \Xi$ . We say that  $f$  satisfies the  $(g, S)$ -Malgrange condition over  $U$  if there exists a constant  $R > 0$  such that

- (1)  $\nabla g(x) \neq 0$  on  $D_{(U,R)} := f^{-1}(U) \setminus \{x \in \mathbb{R}^n \mid R \geq \|x\|\}$
- (2)  $D_{(U,R)} \subset g^{-1}(S)$
- (3)  $\forall_{s \in S} \exists_{\delta_s > 0} \|\nabla_g f(x)\| \|x\| \geq \delta_s \quad \text{for } x \in D_{(U,R)} \cap g^{-1}(s).$

We say that  $f$  satisfies the  $(g, S)$ -Malgrange condition in  $\lambda \in \mathbb{R}$  if there exists neighborhood  $U$  of  $\lambda$  such that  $f$  satisfies the  $(g, S)$ -Malgrange condition over  $U$ . We will denote by  $K_\infty^{(g,S)}(f)$  the set of all  $\lambda$  that don't satisfy the  $(g, S)$ -Malgrange condition.

The main result of this section is the following

**Theorem 3.1.** *Let  $\lambda$  be a regular value of  $f$ . If  $f$  satisfies the  $(g, S)$ -Malgrange condition in  $\lambda \in \mathbb{R}$  then there exists a neighborhood  $U$  of  $\lambda$  such that  $f|_{f^{-1}(U)}$  is a trivial fibration.*

The proof of the above theorem will be preceded by two properties and a lemma.

From now we will assume that  $\lambda \in \mathbb{R}$  is a regular value of  $f$  and that  $f$  satisfies the  $(g, S)$ -Malgrange condition in  $\lambda$ .

The following property holds

**Property 3.2.** *There exists a neighborhood  $U$  of  $\lambda$  such that  $\nabla f(x) \neq 0$  for  $x \in f^{-1}(U)$ .*

Let  $U, R$  be as in the  $(g, S)$ -Malgrange condition. Shrinking the set  $U$  if necessary, we can assume that  $\nabla f(x) \neq 0$  for  $x \in f^{-1}(U)$ . Let  $\alpha, \beta$  be  $C^\infty$  functions in  $\mathbb{R}^n$  such that

$$\alpha(x) = \begin{cases} 0 & \text{for } \|x\| \geq R+1 \\ 1 & \text{for } \|x\| \leq R \end{cases}$$

$$\beta(x) = \begin{cases} 1 & \text{for } \|x\| \geq R+1 \\ 0 & \text{for } \|x\| \leq R \end{cases}$$

and  $0 < \alpha(x), \beta(x) < 1$  for  $\|x\| \in (R, R+1)$ .

We define a smooth vector field  $w : f^{-1}(U) \rightarrow \mathbb{R}^n$  as

$$w(x) := \alpha(x)\nabla f(x) + \beta(x)\nabla_g f(x).$$

Here we are using convention that  $\beta(x)\nabla_g f(x) = 0$  for  $\|x\| \leq R$  (note that  $\nabla_g f(x)$  might not be defined for some points  $x$  such that  $\|x\| \leq R$ ).

From the definition of  $w$  and Property 3.2 we get

**Property 3.3.** *Under the above assumptions we have*

- (i)  $w(x) = \nabla f(x)$  for  $\|x\| \leq R$  and  $w(x) = \nabla_g f(x)$  for  $\|x\| \geq R+1$
- (ii)  $\langle w(x), \nabla f(x) \rangle \neq 0$  for  $x \in f^{-1}(U)$ .

Let us define  $u : f^{-1}(U) \rightarrow \mathbb{R}^n$  as

$$u(x) := \frac{w(x)}{\langle w(x), \nabla f(x) \rangle}.$$

From the assumptions and Property 3.3 we see that  $u$  is well defined, it is smooth and  $u(x) \neq 0$  for  $x \in f^{-1}(U)$ .

**Lemma 3.4.** *Under the above assumptions we have*

- (i)  $\langle \nabla f(x), u(x) \rangle = 1$  for  $x \in f^{-1}(U)$
- (ii) For any  $s \in S$  there exists a constant  $\alpha_s > 0$  such that
 
$$|\langle x, u(x) \rangle| \leq \varepsilon_s \|x\|^2 \quad \text{for } x \in f^{-1}(U) \cap g^{-1}(s), \|x\| > R+1.$$

*Proof.* (i) is obvious. We will prove (ii). From the  $(g, S)$ -Malgrange condition we have  $\frac{1}{\|\nabla_g f\|} \leq \frac{1}{\delta_s} \|x\|$  for  $x \in f^{-1}(U) \cap g^{-1}(s)$ ,  $\|x\| > R$ . Since  $\langle \nabla_g f(x), \nabla f(x) \rangle = \|\nabla_g f(x)\|^2$ , using Schwartz inequality we get

$$|\langle x, u(x) \rangle| \leq \|x\| \|u(x)\| = \|x\| \frac{1}{\|\nabla_g f(x)\|} \leq \frac{1}{\delta_s} \|x\|^2 \text{ for } \|x\| > R+1,$$

which completes the proof.  $\square$

We are ready to prove Theorem 3.1.

*Proof.* Let  $U, R, u$  be defined as above. For each  $\mu \in U$ , consider a system of differential equations

$$(3) \quad x' = (\lambda - \mu)u(x)$$

with the right side defined in  $G := \{(t, x) \in \mathbb{R} \times D \mid x \in f^{-1}(U)\}$ . Denote by  $\Phi_\mu : V_\mu \rightarrow D$  the general solution of system (3) and put  $V_\mu := \{(\tau, \eta, t) \in \mathbb{R} \times D \times R \mid (\tau, \eta) \in G, t \in I_\mu(\tau, \eta)\}$ , where  $I_\mu(\tau, \eta)$  is a domain of integral solution of  $t \rightarrow \Phi_\mu(\tau, \eta, t)$ . From the definition of the general solution we get

$$(4) \quad \Phi_\mu(\tau, \eta, \tau) = \eta.$$

Consider the mapping

$$\Psi_1 : f^{-1}(U) \ni x \mapsto \Phi_{f(x)}(0, x, 1) \in f^{-1}(\lambda).$$

We show that the mapping  $\Psi_1$  is well defined i.e.  $1 \in I_{f(x)}(0, x)$  for each  $x \in f^{-1}(U)$ . Suppose the contrary that there exists  $x \in f^{-1}(U)$  such that  $1 \notin I_{f(x)}(0, x)$ . Then the right end-point  $\beta$  of the interval  $I_{f(x)}(0, x)$  satisfies  $0 < \beta \leq 1$ . Let  $\varphi_x$  be an integral solution of the system (3) with  $\mu = f(x)$  satisfying the initial condition  $\varphi_x(0) = x$ , that is

$$(5) \quad \varphi(t) = \Phi_{f(x)}(0, x, t) \quad \text{for } t \in I_{f(x)}(0, x).$$

We have

$$(f \circ \varphi_x)'(t) = \langle \nabla f(\varphi_x(t)), \varphi_x'(t) \rangle = \lambda - f(x) \quad \text{for } t \in I_{f(x)}(0, x).$$

Therefore

$$(6) \quad f \circ \varphi_x(t) = (\lambda - f(x))t + f(x), \quad t \in I_{f(x)}(0, x)$$

and  $f \circ \varphi_x(t) \in J$  for  $t \in [0, \beta]$ , where  $J$  is a closed interval with endpoints  $\lambda$  and  $f(x)$ .

Denote

$$K := \{(t, x') \in \mathbb{R} \times f^{-1}(U) \mid t \in [0, 1], f(x') \in J, \|x'\| \leq R + 1\}.$$

Obviously  $K$  is a compact set. Lemma 1.3 implies that there exists  $\tau \in (0, \beta)$  such that  $(t, \varphi_x(t)) \notin K$  for  $t \in [\tau, \beta]$ . Since  $J \subset U$ , we have  $\|\varphi_x(t)\| > R + 1$  for  $t \in [\tau, \beta]$ .

Consider a function  $\varrho : [\tau, \beta] \rightarrow \mathbb{R}$  defined by

$$\varrho(t) := \frac{1}{2} \ln \|\varphi_x(t)\|^2.$$

Let  $s_0 \in S$  be such that  $\varphi_x(\tau) \in D_{s_0}$ . Using Lemma 3.4 (ii) we get

$$|\varrho'(t)| = \frac{|\langle \varphi_x(t), \varphi'_x(t) \rangle|}{\|\varphi_x(t)\|^2} = \frac{|\lambda - f(x)|}{\|\varphi_x(t)\|^2} |\langle \varphi_x(t), u(\varphi_x(t)) \rangle| \leq \varepsilon_{s_0} |\lambda - f(x)|$$

for  $t \in (\tau, \beta)$ . From the mean value theorem there exists  $\theta_t \in (\tau, t)$  such that  $\varrho(t) - \varrho(\tau) = \varrho'(\theta_t)(t - \tau)$  and therefore, by the above,

$$\varrho(t) \leq \varrho(\tau) + \varepsilon_{s_0} |\lambda - f(x)| (t - \tau) \leq \varrho(\tau) + \varepsilon_{s_0} |\lambda - f(x)| (\beta - \tau).$$

Denoting  $L := \varrho(\tau) + \varepsilon_{s_0} |\lambda - f(x)| (\beta - \tau)$  we see that the solution  $\varphi_x|_{(\tau, \beta)}$  is contained in the compact set

$$\{(t, x') \in \mathbb{R} \times f^{-1}(U) \mid f(x') \in J, \|x'\| \leq e^L\} \subset \mathbb{R} \times f^{-1}(U),$$

which contradicts Lemma 1.3.

Summing up we have shown that  $1 \in I_{f(x)}(0, x)$  for every  $x \in f^{-1}(U)$ . Then from (6) we get  $f(\Psi_1(x)) = f(\varphi_x(1)) = \lambda$  and the mapping  $\Psi_1$  is defined correctly. Similarly as above we show that the mapping

$$\Theta : f^{-1}(\lambda) \times U \ni (\xi, \mu) \mapsto \Phi_\mu(1, \xi, 0) \in f^{-1}(U)$$

is also well defined. It is easy to check that the mapping

$$\Psi : f^{-1}(U) \ni x \mapsto (\Psi_1(x), f(x)) \in f^{-1}(\lambda) \times U$$

is a  $C^\infty$  diffeomorphism and  $\Psi^{-1} = \Theta$ . Therefore  $f|_{f^{-1}(U)}$  is a trivial fibration.  $\square$

Immediately from Theorem 3.1 we get

**Remark 3.5.** For every  $(g, S) \in \Xi$  we have  $B(f) \subset K_\infty^{(g, S)}(f) \cup K_0(f)$ . Therefore  $B(f) \subset \bigcap_{(g, S) \in \Xi} K_\infty^{(g, S)}(f) \cup K_0(f)$ .

In Theorem 3.1 we showed how to trivialize a function  $f$  using fibers of some function  $g$ . Now we give some examples where the assumptions of Theorem 1 are not met but using Theorem 3.1 we can deduce the triviality of  $f$ .

**Example 3.6.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) := \frac{y}{1 + x^2}.$$

It is easy to check that  $f$  does not satisfy the Malgrange condition in 0. Denote  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$g(x, y) := x, \quad S = \mathbb{R}.$$

Put  $U = \mathbb{R}$  and  $R = 1$ . Then

- (1)  $\nabla g(x) = [1, 0] \neq 0$  for  $D_{(U, R)} = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 \mid R \geq \|x\|\}$
- (2)  $D_{(U, R)} \subset g^{-1}(\mathbb{R})$



(3) for  $s \in \mathbb{R}$  we have

$$\|\nabla_g f(x, y)\| = \|[0, \frac{1}{1+x^2}]\| = \frac{1}{1+s^2} \quad \text{for } (x, y) \in D_{(U,R)} \cap g^{-1}(s).$$

Therefore  $f$  satisfies the  $(g, S)$ -Malgrange condition and using Theorem 3.1 we deduce that  $f$  is a trivial fibration.

An example of polynomials which is a trivial fibration but does not satisfy the Malgrange condition comes from L. Păunescu and A. Zaharia (see [PZ]).

**Example 3.7.** Let  $p, q \in \mathbb{N}$ ,  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$f(x, y, z) := x - 3x^{2p+1}y^{2q} + 2x^{3p+1}y^{3q} + yz.$$

L. Păunescu and A. Zaharia showed that after a suitable polynomial change of coordinates, we can write  $f(X, y, Z) = X$ . Therefore  $f$  is a trivial fibration. Following their reasoning, we can deduce that if  $p > q$  then  $f$  does not satisfy the Malgrange condition in 0. Let

$$g(x, y, z) := y \quad \text{for } (x, y, z) \in \mathbb{R}^3, S = \mathbb{R}.$$

Put  $U = \mathbb{R}$ ,  $R = 1$ . Then

- (1)  $\nabla g(x, y, z) = [0, 1, 0] \neq 0$  for  $D_{(U,R)} = \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3 \mid R \geq \|x\|\}$
- (2)  $D_{(U,R)} \subset g^{-1}(\mathbb{R})$
- (3) for  $s \in \mathbb{R}$  we have

$$\begin{aligned} \nabla_g f(x, y, z) &= [1 - 3(2p+1)x^{2p}y^{2q} + 2(3p+1)x^{3p}y^{3q}, 0, y] \\ &= [1 - 3(2p+1)x^{2p}s^{2q} + 2(3p+1)x^{3p}s^{3q}, 0, s] \end{aligned}$$

for  $(x, y) \in D_{(U,R)} \cap g^{-1}(s)$ . If  $s \neq 0$  then  $\|\nabla_g f(x, y, z)\| \geq \|s\| > 0$  and if  $s = 0$  we have  $\|\nabla_g f(x, y, z)\| = 1$ .

Therefore  $f$  satisfies the  $(g, S)$ -Malgrange condition and using Theorem 3.1 we deduce that  $f$  is a trivial fibration.

In general, finding a suitable function  $g$  can be very difficult. In the case when  $f$  is a coordinate of a mapping with non-vanishing jacobian, the natural candidates for  $g$  are other coordinates of this mapping.

**Example 3.8.** Let  $F = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $Jac(F) = 1$ , where  $Jac(F)$  is the jacobian of  $F$ . Than we have

$$\|\nabla_{f_2} f_1(x, y)\|^2 = \frac{Jac(F)^2}{\|\nabla f_2(x, y)\|^2} = \frac{1}{\|\nabla f_2(x, y)\|^2}, \quad (x, y) \in \mathbb{R}^2.$$

Therefore  $f_1$  satisfies the  $(f_2, \mathbb{R})$ -Malgrange condition over  $\mathbb{R}$  if and only if there exists  $R > 0$  such that

$$\|(x, y)\| > \delta_s \|\nabla f_2(x, y)\| \quad \text{for } (x, y) \in f_2^{-1}(s), \|(x, y)\| > R, s \in \mathbb{R}.$$

From Example 3.8 we get

**Remark 3.9.** Let  $F = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a smooth mapping with  $Jac(F) = 1$  and  $f_2$  be a polynomial, such that  $\deg f_2 \leq 2$ . Then the mapping  $F$  is injective.

#### 4. $\rho_0$ -REGULARITY ON MANIFOLDS

In this section we will consider a different condition that allows us to trivialize functions defined on the manifold of the form  $g^{-1}(0)$  at infinity.

Let  $f, g \in C^\infty(\mathbb{R}^n)$ . Assume that  $\nabla g(x) \neq 0$  for  $x \in g^{-1}(0)$  and denote  $M := g^{-1}(0)$ ,  $\rho_0 := \|\cdot\|^2|_M$ .

The critical set  $\mathcal{M}_0(f_M)$  of the map  $(f_M, \rho_0) : M \rightarrow \mathbb{R}$  we will call *the Milnor set of  $f_M$*  (with respect to  $\rho_0$  function).

A value  $\lambda \in \mathbb{R}$  is called a  $\rho_0$ -regular value of  $f_M$  at infinity if there exist a neighborhood  $U$  of  $\lambda$  and a constant  $R > 0$  such that

$$\forall_{x \in f_M^{-1}(U)} \quad x \notin \mathcal{M}_0(f_M) \quad \text{for} \quad \|x\| \geq R.$$

The set  $S_0(f_M)$  of all values that are not a  $\rho_0$  regular value of  $f_M$  at infinity will be called the set of *asymptotic  $\rho_0$ -nonregular values* of  $f_M$  i.e.

$$S_0(f_M) := \{\lambda \in \mathbb{R} \mid \exists_{(x_k) \subset \mathcal{M}_0(f_M)} \|x_k\| \rightarrow \infty, f_M(x_k) \rightarrow \lambda\}.$$

Our aim is to prove the following theorem

**Theorem 4.1.** *If  $\lambda$  is a  $\rho_0$ -regular value of  $f_M$  at infinity then there exists neighborhood  $U$  of the  $\lambda$  and  $R > 0$  such that  $f_M|_{f_M^{-1}(U)}$  is a  $C^\infty$  trivial fibration on  $M$  at infinity.*

The proof of the theorem will be preceded by some technical properties.

Let  $M^* := M \setminus \mathcal{M}_0(f_M)$  and consider the vector field  $v : M^* \rightarrow \mathbb{R}^n$

$$\begin{aligned} v(x) := & \nabla f(x) + \frac{\langle \nabla g(x), x \rangle \langle \nabla g(x), \nabla f(x) \rangle - \|\nabla g(x)\|^2 \langle x, \nabla f(x) \rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2} x \\ & + \frac{\langle \nabla g(x), x \rangle \langle x, \nabla f(x) \rangle - \|x\|^2 \langle \nabla g(x), \nabla f(x) \rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2} \nabla g(x) \text{ for } x \in M^*. \end{aligned}$$

The field  $v$  is well defined. Indeed if  $\|x\| \|\nabla g(x)\| = |\langle \nabla g(x), x \rangle|$  then using the Cauchy–Schwarz inequality we deduce that  $x$  and  $\nabla g(x)$  are linearly dependent. We get  $\nabla \rho_0(x) = 0$  and therefore  $x \in \mathcal{M}_0(f_M)$  which contradicts the assumptions.

From the definition of  $v$  we see that  $v(x)$  is tangent to the manifold  $M^*$  and to the sphere  $\partial B(x) := \{y \in \mathbb{R}^n \mid \|y\| = \|x\|\}$ . Namely, we have

**Property 4.2.** *For any  $x \in M^*$  we have  $\langle v(x), x \rangle = \langle v(x), \nabla g(x) \rangle = 0$ .*

*Proof.* Take  $x \in M^*$ . Then

$$\begin{aligned} \langle v(x), x \rangle &= \langle \nabla f(x), x \rangle \\ &+ \frac{-\|\nabla g(x)\|^2 \langle x, \nabla f(x) \rangle \|x\|^2 + \langle \nabla g(x), x \rangle^2 \langle x, \nabla f(x) \rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2}, \end{aligned}$$

therefore

$$\begin{aligned} \langle v(x), x \rangle &= \langle \nabla f(x), x \rangle \\ &+ \langle \nabla f(x), x \rangle \frac{-\|\nabla g(x)\|^2 \|x\|^2 + \langle \nabla g(x), x \rangle^2}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2} = 0, \end{aligned}$$

which gives that  $\langle v(x), x \rangle = 0$ . Analogously as above we obtain that  $\langle v(x), \nabla g(x) \rangle = 0$ .  $\square$

**Lemma 4.3.** *For  $x \in M^*$  we have  $\langle v(x), \nabla f_M(x) \rangle \neq 0$ .*

*Proof.* Denote

$$\begin{aligned} \Omega &:= \{w : M^* \rightarrow \mathbb{R}^n \mid w - \text{smooth}\}, \\ \Omega(TM^*) &:= \{w \in \Omega \mid \forall_{x \in M^*} \langle w(x), \nabla g(x) \rangle = 0\} \end{aligned}$$

and let  $\pi_{M^*} : \Omega \rightarrow \Omega(TM^*)$  be a mapping defined by

$$\pi_{M^*}(w(x)) := w(x) - \frac{\langle w(x), \nabla g(x) \rangle}{\|\nabla g(x)\|^2} \nabla g(x) \quad \text{for } x \in M^*.$$

At first we will prove that

$$(7) \quad v(x) = \pi_{(\pi_{M^*}(x))^\perp}(\nabla f_M(x)) \quad \text{for } x \in M^*,$$

where

$$\pi_{(\pi_{M^*}(x))^\perp}(w(x)) := w(x) - \frac{\langle w(x), \pi_{M^*}(x) \rangle}{\|\pi_{M^*}(x)\|^2} \pi_{M^*}(x) \quad \text{for } x \in M^*, w \in \Omega.$$

From definitions and simple calculations for  $x \in M^*$  we have

$$\begin{aligned} \pi_{(\pi_{M^*}(x))^\perp}(\nabla f_M(x)) &= \nabla f_M(x) - \frac{\langle \nabla f_M(x), \pi_{M^*}(x) \rangle}{\|\pi_{M^*}(x)\|^2} \pi_{M^*}(x) = \\ &= \nabla f(x) - \frac{\langle \nabla f(x), \nabla g(x) \rangle}{\|\nabla g(x)\|^2} \nabla g(x) - \frac{\langle \nabla f(x), \pi_{M^*}(x) \rangle}{\|\pi_{M^*}(x)\|^2} \pi_{M^*}(x) = \\ &= \nabla f(x) + \frac{\langle \nabla g(x), x \rangle \langle \nabla g(x), \nabla f(x) \rangle - \|\nabla g(x)\|^2 \langle x, \nabla f(x) \rangle}{\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2} x + \\ &\quad + \frac{\langle \nabla f(x), \|\nabla g(x)\|^2 x - \langle \nabla g(x), x \rangle \nabla g(x) \rangle \langle \nabla g(x), x \rangle}{\|\nabla g(x)\|^2 (\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2)} \nabla g(x) + \\ &\quad - \frac{\langle \nabla f(x), \nabla g(x) \rangle (\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2)}{\|\nabla g(x)\|^2 (\|x\|^2 \|\nabla g(x)\|^2 - \langle \nabla g(x), x \rangle^2)} \nabla g(x) = v(x). \end{aligned}$$

which proves (7).

For  $x \in M^*$  we have

$$\begin{aligned} \langle v(x), \nabla f_M(x) \rangle &= 0 \\ \Leftrightarrow \langle \pi_{(\pi_{M^*}(x))^\perp}(\nabla f_M(x)), \nabla f_M(x) \rangle &= 0 \\ \Leftrightarrow \|\nabla f_M(x)\|^2 \|\pi_{M^*}(x)\|^2 &= \langle \nabla f_M(x), \pi_{M^*}(x) \rangle^2 \\ \Rightarrow x &\in \mathcal{M}_0(f_M) \end{aligned}$$

which completes the proof.  $\square$

**Remark 4.4.** Analogously as in the proof of Lemma 4.3 we can prove that  $v(x) = \pi_{(\pi_{\partial B}(\nabla g(x)))^\perp}(\pi_{\partial B}(\nabla f(x)))$ , where

$$\begin{aligned} \pi_{\partial B}(w(x)) &:= w(x) - \frac{\langle w(x), x \rangle}{\|x\|^2} x, \\ \pi_{(\pi_{\partial B}(\nabla g(x)))^\perp}(w(x)) &:= w(x) - \frac{\langle w(x), \pi_{\partial B}(\nabla g(x)) \rangle}{\|\pi_{\partial B}(\nabla g(x))\|^2} \pi_{\partial B}(\nabla g(x)) \end{aligned}$$

for  $x \in M^*, w \in \Omega$ .

We are ready to prove Theorem 4.1.

*Proof.* By the assumption that  $\lambda$  is a  $\rho_0$ -regular value of  $f_M$  at infinity, there exist a neighborhood  $U$  of the  $\lambda$  and  $R > 0$  such that

$$(\mathcal{M}_0(f_M) \cap f_M^{-1}(U)) \setminus \overline{B(R)} = \emptyset,$$

where  $\overline{B(R)} := \{x \in M \mid R \geq \|x\|\}$ . Let  $w(x) := \frac{v(x)}{\langle v(x), f_M(x) \rangle}$  for  $x \in f_M^{-1}(U) \setminus \overline{B(R)}$ . From the assumption and Lemma 4.3,  $w$  is well defined. For each  $\mu \in U$  consider the following system of differential equations

$$(8) \quad x' = (\lambda - \mu)w(x)$$

with the right hand side defined in the set  $G := \{(t, x) \in \mathbb{R} \times M \mid x \in f_M^{-1}(U) \setminus \overline{B(R)}\}$ . Denote by  $\Phi_\mu : V_\mu \rightarrow M$  the general solution of system (8) and  $V_\mu := \{(\tau, \eta, t) \in \mathbb{R} \times M \times R \mid (\tau, \eta) \in G, t \in I_\mu(\tau, \eta)\}$ , where  $I_\mu(\tau, \eta)$  is a domain of integral solution of  $t \rightarrow \Phi_\mu(\tau, \eta, t)$ . From a definition of the general solution we get

$$(9) \quad \Phi_\mu(\tau, \eta, \tau) = \eta.$$

Note that Property 4.2 implies

$$(10) \quad \|\Phi_\mu(\tau, \eta, t)\| = \|\eta\| \text{ for } t \in I_\mu(\tau, \eta), (\tau, \eta) \in G, \mu \in U.$$

Consider the mapping

$$\Psi_1 : f_M^{-1}(U) \setminus \overline{B(R)} \ni x \mapsto \Phi_{f(x)}(0, x, 1) \in f_M^{-1}(\lambda) \setminus \overline{B(R)}.$$

We show the mapping  $\Psi_1$  is well defined that is  $1 \in I_{f(x)}(0, x)$  for each  $x \in f_M^{-1}(U) \setminus \overline{B(R)}$ .

Suppose the contrary that there exists  $x \in f_M^{-1}(U) \setminus \overline{B(R)}$  such that  $1 \notin I_{f(x)}(0, x)$  and denote  $\varphi_x(t) := \Phi_{f(x)}(0, x, t)$  for  $t \in I_{f(x)}(0, x)$ . From (8) and the initial condition (9) we get

$$(11) \quad f_M \circ \varphi_x(t) = (\lambda - f(x))t + f(x), \quad t \in I_{f(x)}(0, x).$$

Denoting  $J$  as closed interval with endpoints  $\lambda$  and  $f(x)$  we see that the set

$$F := \{(t, x') \in \mathbb{R} \times f_M^{-1}(U) \setminus \overline{B(R)} \mid t \in [0, 1], f_M(x') \in J, \|x'\| = \|x\|\}.$$

is a compact subset of  $G$  such that the graph of  $\varphi_x|_{[0, \beta)}$  is contained in  $F$ . This contradicts Lemma 1.3 and proves  $[0, 1] \subset I_{f(x)}(0, x)$ .

Using (11) we get  $f(\Psi_1(x)) = f(\varphi_x(1)) = \lambda$  and from (10) we have  $\Psi_1(x) \in f_M^{-1}(\lambda) \setminus \overline{B(R)}$ . Summing up we show that the mapping  $\Psi_1$  is defined correctly. Similarly we can show that the mapping

$$\Theta : f_M^{-1}(\lambda) \setminus \overline{B(R)} \times U \ni (\xi, \mu) \mapsto \Phi_\mu(1, \xi, 0) \in f_M^{-1}(U) \setminus \overline{B(R)}$$

is well defined. It is easy to check that the mapping

$$\Psi : f_M^{-1}(U) \setminus \overline{B(R)} \ni x \mapsto (\Psi_1(x), f_M(x)) \in f_M^{-1}(\lambda) \setminus \overline{B(R)} \times U$$

is a  $C^\infty$  diffeomorphism and  $\Psi^{-1} = \Theta$ . Therefore  $f_M|_{f_M^{-1}(U) \setminus \overline{B(R)}}$  is a  $C^\infty$  trivial fibration on  $M$ .  $\square$

**Remark 4.5.** From Theorem 4.1 we have  $B_\infty(f) \subset S_0(f)$ .

It is worth noting that unlike in a flat case we need to take into account the set  $V := \{x \in M \mid \exists_{t \in \mathbb{R}} \nabla g(x) = tx\} \subset \mathcal{M}_0(f_M)$ . In general the field  $v : M \setminus V \rightarrow \mathbb{R}^n$  can not be continuously extended on the set  $V$ . The following example illustrates the fact

**Example 4.6.** Let

$$g(x, y, z) := \frac{1}{2}x^2 + y^2 - \frac{1}{2}, \quad f(x, y, z) = y \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

We have

$$\begin{aligned} v(x, y, z) &= [v^1(x, y, z), v^2(x, y, z), v^3(x, y, z)] \\ &= \left[ \frac{-2xyz^2}{x^2y^2 + x^2z^2 + 4y^2z^2}, \frac{x^2z^2}{x^2y^2 + x^2z^2 + 4y^2z^2}, \frac{x^2yz}{x^2y^2 + x^2z^2 + 4y^2z^2} \right] \end{aligned}$$

for  $(x, y, z) \in M \setminus V$ . Note that  $(1, 0, 0) \in V$  and

$$v^2(1, 0, z) = 1 \text{ for } z \neq 0 \text{ and } v^2(x, y, 0) = 0 \text{ for } x \neq 0, y \neq 0,$$

therefore the limit  $\lim_{(x, y, z) \rightarrow (1, 0, 0)} v^2(x, y, z)$  does not exist.

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